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Excited Two-Mode Generalized Squeezed Vacuum State as a Squeezed Two-Variable Hermite Polynomial Excitation State

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Abstract A new class of excited two-mode generalized squeezed vacuum states denoted by $|r, s, m, n\rangle$ are presented, which are obtained by repeatedly applying creation operators $a^\dagger$ and $b^\dagger$ on the two-mode generalized squeezed vacuum state. We find that it is just regarded as a generalized squeezed two-variable Hermite polynomial excitation on the vacuum state and its normalization constant is just a Jacobi polynomial. Their statistical properties are investigated such as squeezing properties, photon number distribution and the violations of Cauchy-Schwartz inequality. Especially, the Wigner function for $|r, s, m, n\rangle$ depending on the excitation photon numbers is discussed graphically.

Keywords Two-mode generalized squeezed vacuum state · Two-variable Hermite polynomial · Violations of Cauchy-Schwartz inequality · Wigner function

1 Introduction

Squeezed states and squeezing operators have been an important topic in quantum optics due to their potential applications. Recently, more and more groups have devoted to researching on well-behavior nonclassical properties of squeezed states [1–3], especially for the two-mode squeezed states [4–11]. For example, phase properties for the two-mode squeezed states have been widely investigated using the Pegg-Barnett phase formalism in [5–7]. Hiroshima also study the decoherence of two-mode squeezed vacuum states by analyzing the relative entropy of entanglement [8]. The two-mode squeezed-vacuum state is regarded as

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quantum channel to realize quantum teleportation and quantum dense coding [9], respectively. Recently, Hu and Fan [10] report that two-mode squeezed number state is considered as the two-mode squeezed vacuum state excited by the two-variable Hermite polynomial. In addition, some generalized two-mode squeezed states have also been constructed, which are great progress in quantum optics [12–14].

On the other hand, a class of nonclassical states, generated by applying appropriate operators on certain states, have also attracted much attention, such as excited coherent state (photon-added coherent state) [15, 16], excited single-mode squeezed vacuum state [17, 18], excited pair coherent state [19, 20], excited entangled coherent state [21, 22], and so on. Because these states are of particular interest since they exhibit quite significant and unusual physical properties, the study of which can not only deepen our understanding of the nature of quantum fields but also help us to realize these fields experimentally for possible future applications such as in weak-signal detection, quantum computing, and quantum coding.

In present paper, enlightened by the above works, we shall construct a new state named as the excited two-mode generalized squeezed vacuum states (ETGSVSs), denoted by $|r,s,m,n\rangle$, which are obtained by repeatedly applying creation operators $a^{\dagger}$ and $b^{\dagger}$ on the two-mode generalized squeezed vacuum state $U_2(r,s)|0,0\rangle$, i.e.

$$|r,s,m,n\rangle = N_{s,m,n} a^{\dagger m} b^{\dagger n} U_2(r,s)|0,0\rangle \tag{1}$$

where $N_{s,m,n}$ is normalization factor, $|0,0\rangle$ is two-mode vacuum state, and $U_2(r,s)$ (r, s are complex numbers, $|s|^2-|r|^2=1$) is generalized squeezed operator with three parameters [23, 24]. Our main aim is to explore theoretically the statistical properties of the ETGSVSs. This paper is arranged as follows. In Sect. 2, we introduce the ETGSVSs, and then its normalization constant is derived, which turns out to be a Jacobi polynomial. It is found that the ETGSVSs is just regarded as a generalized squeezed two-variable Hermite polynomial excitation on the vacuum state. In Sect. 3 the statistical properties of the ETGSVSs are investigated, such as squeezing properties, photon number distribution. In addition, the violations of Cauchy-Schwartz inequality is evaluated as well. In Sect. 4 we focus our attention on calculating the Wigner function for the ETGSVSs by using the Weyl ordered operators' invariance under similar transformations, which is proportional to the two-variable Gaussian-Hermite polynomials. Its behaviors depending on the excitation photon numbers are studied by virtue of drawing the three-dimensional graphics. In Sect. 5 we conclude our letter with a summary and some discussion.

2 ETGSVSs as a Squeezed Two-Variable Hermite Polynomial Excitation State

To begin with, based on the definition in (1), we derive the normalization constant of the ETGSVSs in more detail. For this purpose, we recall the generalized two-mode squeezed operator $U_2(r,s)$ with three parameters.

As is described by [23, 24], the form of $U_2(r,s)$ in the two-mode coherent state $|z_1,z_2\rangle$ representation is

$$U_2(r,s) = s\int \frac{d^2z_1 d^2z_2}{\pi^2}|sz_1+rz_2^*, rz_1^*+sz_2\rangle\langle z_1,z_2|, \tag{2}$$

where s and r are complex and satisfy the unimodularity condition $|s|^2-|r|^2=1$, $(z_1,z_2)\to(sz_1+rz_2^*, rz_1^*+sz_2)$ is the mapping of symmetric transform in (z_1,z_2) phase space. Further, by virtue of the technique of integral within an ordered product of operators technique [25], it can be recast as

$$U_2(r,s)=\exp\left(\frac{r}{s^*}a^\dagger b^\dagger\right)\exp\left[(a^\dagger a+b^\dagger b+1)\ln(s^*)^{-1}\right]\exp\left(-\frac{r^*}{s^*}ab\right), \tag{3}$$

where we have utilized the normal ordering of two-mode vacuum projector

$$|0,0\rangle\langle 0,0| =:\exp(-a^\dagger a-b^\dagger b):, \tag{4}$$

and the operator identity

$$\exp(\gamma a^\dagger a) =:\exp[(e^\gamma-1)a^\dagger a]:. \tag{5}$$

It is seen from (3) that when $s=\cosh\lambda$ and $r=\sinh\lambda$, $U_2(r,s)$ is back to the usual two-mode squeezed operator [1, 2]. So $U_2(r,s)$ is called the generalized two-mode squeezed operator with three parameters.

The normally ordered form (3) is appropriate for proving the following unitary two-mode squeezing transform

$$\begin{aligned} &U_2^\dagger(r^*,s^*)aU_2(r,s)=sa+rb^\dagger, \qquad && U_2^\dagger(r^*,s^*)a^\dagger U_2(r,s)=s^*a^\dagger+r^*b,\\ &U_2^\dagger(r^*,s^*)bU_2(r,s)=ra^\dagger+sb, \qquad && U_2^\dagger(r^*,s^*)b^\dagger U_2(r,s)=r^*a+s^*b^\dagger. \end{aligned} \tag{6}$$

Based on the above squeezing transformation in (6), the normalization constant $N_{r,s,m,n}$ can be derived as follows

$$\begin{aligned} N_{s,m,n}^{-2}&=|s|^{2n}n!\langle 0,n|(sa+rb^\dagger)^m(s^*a^\dagger+r^*b)^m|0,n\rangle\\ &=|s|^{2n}n!\sum_{k,l=0}^{\min(m,n)}\frac{m!m!}{k!(m-k)!l!(m-l)!}\langle 0,n|(sa)^{m-k}(rb^\dagger)^k(s^*a^\dagger)^{m-l}(r^*b)^l|0,n\rangle\\ &=|s|^{2m}m!n!|s|^{2n}\sum_{k=0}^{\min(m,n)}\frac{m!n!}{(k!)^2(m-k)!(n-k)!}\left|\frac{r}{s}\right|^{2k}. \end{aligned} \tag{7}$$

Without lossing of generality, supposing $m>n$ and comparing (7) with the standard expression of Jacobi polynomials [26]

$$P_n^{(\alpha,\beta)}(x)=\left(\frac{x-1}{2}\right)^n\sum_{k=0}^{n}\frac{(n+\alpha)!}{k!(n+\alpha-k)!}\frac{(n+\beta)!}{(n-k)!(\beta+k)!}\left(\frac{x+1}{x-1}\right)^k, \tag{8}$$

(8) can be rewritten into

$$P_n^{(m-n,0)}(1-2|s|^2)=(-|s|^2)^n\sum_{k=0}^{n}\frac{m!}{k!(m-k)!}\frac{n!}{(n-k)!k!}\left|\frac{r}{s}\right|^{2k}. \tag{9}$$

Substituting (9) into (7), one has

$$\begin{aligned} N_{s,m,n}^{-2}&=|s|^{2m}m!n!(-1)^nP_n^{(m-n,0)}(1-2|s|^2)\\ &=|s|^{2m}m!n!P_n^{(0,m-n)}(2|s|^2-1), \end{aligned} \tag{10}$$

where, in the last step, we have used the properties of Jacobi polynomial

$$(-1)^n P_n^{(m-n,0)}(1-2|s|^2) = P_n^{(0,m-n)}(2|s|^2-1). \tag{11}$$

Similarly, the overlap $\langle r,s,m+u,n+t|$ and $|r,s,m,n\rangle$ can also be calculated

$$\langle r,s,m+u,n+t|r,s,m,n\rangle = N_{s,m,n}^2 |s|^{2m} n!(m+u)!(rs)^u \delta_{t,u} P_n^{(u,m-n)}(2|s|^2-1). \tag{12}$$

On the other hand, by employing (3) we can easily reduce (1) to

$$|r,s,m,n\rangle \equiv N_{s,m,n} a^{\dagger m} b^{\dagger n} U_2(r,s)|0,0\rangle = N_{s,m,n}(s^*)^{-1} a^{\dagger m} b^{\dagger n} \exp\left(\frac{r}{s^*} a^\dagger b^\dagger\right)|0,0\rangle. \tag{13}$$

Furthermore, note that $a^\dagger|0\rangle = \sqrt{n!}|n\rangle$ and $a^k|n\rangle = \sqrt{n!}/(n-k)! a^{\dagger n-k}|0\rangle$, which directly lead to $a^k a^{\dagger n}|0,0\rangle = n!/(n-k)! a^{\dagger n-k}|0\rangle$, thus (1) can be re-expressed as

$$\begin{aligned}
|r,s,m,n\rangle &= N_{s,m,n}(-i)^{m+n}(r^* s^*)^n U_2(r,s) \\
&\quad \times \sum_{k=0}^{\min(m,n)} \frac{m!n!(-1)^k}{k!(m-k)!(n-k)!} (is^* a^\dagger)^{m-k} \left(i\frac{b^\dagger}{r^*}\right)^{n-k} |0,0\rangle \\
&= N_{s,m,n}(-i)^{m+n}(r^* s^*)^n U_2(r,s) H_{m,n}\left(is^* a^\dagger, i\frac{b^\dagger}{r^*}\right)|0,0\rangle,
\end{aligned} \tag{14}$$

where the definition of the two variables Hermitian polynomial [27],

$$H_{m,n}(\zeta,\eta) = \sum_{k=0}^{\min(m,n)} \frac{m!n!(-1)^k}{k!(m-k)!(n-k)!} \zeta^{m-k} \eta^{n-k}, \tag{15}$$

has been used.

Obviously, it then follows from (14) that the ETGSVSs $|r,s,m,n\rangle$ is equivalent to a two-mode squeezed two-variable Hermite-excited vacuum state. It is clearly seen that, when $m=n=0$, $s=\cosh\lambda$ and $r=\sinh\lambda$, (14) just reduces to the usual two-mode squeezed state due to $H_{0,0}=1$.

3 Statistical Properties for ETGSVSs

3.1 Squeezing Properties

In order to analyze the squeezing properties, we first begin with introducing the optical quadrature amplitudes for a two-mode system [1, 2],

$$X = \frac{X_1+X_2}{\sqrt{2}}, \qquad P = \frac{P_1+P_2}{\sqrt{2}}, \quad [X,P]=i, \tag{16}$$

where $X_1=(a+a^\dagger)/2^{1/2}$ ($X_2=(b+b^\dagger)/2^{1/2}$) and $P_1=(a-a^\dagger)/(2^{1/2}i)$ ($P_2=(b-b^\dagger)/(2^{1/2}i)$) are coordinate and momentum operators, respectively. Their variances are $(\Delta X)^2=\langle X^2\rangle-\langle X\rangle^2$, $(\Delta P)^2=\langle P^2\rangle-\langle P\rangle^2$. The fluctuations satisfy the uncertainty relation of quantum mechanics $\Delta X\Delta P\geq 1/2$. Using (13) and the orthogonality of Fock representation, it is readily to see that $\langle a\rangle=\langle a^\dagger\rangle=\langle b\rangle=\langle b^\dagger\rangle=0$, which immediately leads to

$\langle X\rangle = \langle P\rangle = 0$, and $\langle a^2\rangle = \langle a^{\dagger 2}\rangle = \langle b^2\rangle = \langle b^{\dagger 2}\rangle = 0$ as well as $\langle ab^\dagger\rangle = \langle a^\dagger b\rangle = 0$. It then follows that

$$(\Delta X)^2 = \frac{1}{2}(\langle aa^\dagger\rangle + \langle bb^\dagger\rangle + \langle ab\rangle + \langle a^\dagger b^\dagger\rangle - 1). \tag{17}$$

Then, making use of (10), one can readily derive

$$\begin{aligned}\langle aa^\dagger\rangle &= \langle r, s, m, n|aa^\dagger|r, s, m, n\rangle \\ &= |s|^2(m+1)\frac{P_n^{(0,m-n+1)}(2|s|^2-1)}{P_n^{(0,m-n)}(2|s|^2-1)},\end{aligned} \tag{18}$$

$$\langle bb^\dagger\rangle = (n+1)\frac{P_{n+1}^{(0,m-n-1)}(2|s|^2-1)}{P_n^{(0,m-n)}(2|s|^2-1)}. \tag{19}$$

Again, it can be seen by employing (12) that

$$\langle ab\rangle = \frac{(m+1)rs\,P_n^{(1,m-n)}(2|s|^2-1)}{P_n^{(0,m-n)}(2|s|^2-1)}, \tag{20}$$

and

$$\langle a^\dagger b^\dagger\rangle = \frac{(m+1)r^*s^*\,P_n^{(1,m-n)}(2|s|^2-1)}{P_n^{(0,m-n)}(2|s|^2-1)}. \tag{21}$$

It then follows by inserting (18)–(21) into (17) that

$$\begin{aligned}(\Delta X)^2 = {}& \frac{1}{2P_n^{(0,m-n)}(2|s|^2-1)}\big[|s|^2(m+1)\,P_n^{(0,m-n+1)}(2|s|^2-1) \\ &+ (n+1)\,P_{n+1}^{(0,m-n-1)}(2|s|^2-1) + (m+1)rs\,P_n^{(1,m-n)}(2|s|^2-1) \\ &+ (m+1)r^*s^*\,P_n^{(1,m-n)}(2|s|^2-1)\big] - \frac{1}{2},\end{aligned} \tag{22}$$

and

$$\begin{aligned}(\Delta P)^2 = {}& \frac{1}{2P_n^{(0,m-n)}(2|s|^2-1)}\big[|s|^2(m+1)\,P_n^{(0,m-n+1)}(2|s|^2-1) \\ &+ (n+1)\,P_{n+1}^{(0,m-n-1)}(2|s|^2-1) - (m+1)rs\,P_n^{(1,m-n)}(2|s|^2-1) \\ &- (m+1)r^*s^*\,P_n^{(1,m-n)}(2|s|^2-1)\big] - \frac{1}{2}.\end{aligned} \tag{23}$$

Especially, when there is no excitation, i.e., $m = n = 0$, (22) and (23) reduce to

$$(\Delta X)^2|_{m=n=0} = \frac{1}{2}|s+r|^2, \qquad (\Delta P)^2 = \frac{1}{2}|s-r|^2, \qquad \Delta X\Delta P = \frac{1}{2}|s^2-r^2|, \tag{24}$$

which seems to be quite different from the squeezing effect caused by the usual two-mode squeezing operator. Further, setting $s = \cosh\lambda$ and $r = \sinh\lambda$, it is easily seen from (24) that

$$(\Delta X)^2|_{m=n=0} = \frac{1}{2}e^{2\lambda}, \qquad (\Delta P)^2|_{m=n=0} = \frac{1}{2}e^{-2\lambda}, \qquad \Delta X\Delta P = \frac{1}{2},$$

which is just the standard squeezing case.

3.2 Photon Number Distribution

Next we will discuss the photon number distribution in the ETGSVS. We first evaluate the overlap between the two-mode number state $\langle n_a, n_b|$ and $|r, s, m, n\rangle$. Using the expression in (13), one can obtain the overlap

$$\begin{aligned}\langle n_a, n_b|r, s, m, n\rangle &= N_{s,m,n}\langle n_a, n_b|a^{\dagger m}b^{\dagger n}(s^*)^{-1}\exp\left(\frac{r}{s^*}a^\dagger b^\dagger\right)|0,0\rangle \\ &= N_{s,m,n}(s^*)^{-1}\left(\frac{r}{s^*}\right)^{n_a-m}\frac{\sqrt{n_a!n_b!}}{(n_a-m)!}\delta_{n_a-m,n_b-n}.\end{aligned} \tag{25}$$

So we can immediately obtain the explicit form of the photon number distribution in the TEGSVS

$$P(n_a, n_b) = |\langle n_a, n_b|r, s, m, n\rangle|^2 = \frac{|(s^*)^{-1}(\frac{r}{s^*})^{n_a-m}\frac{\sqrt{n_a!n_b!}}{\sqrt{(n_a-m)!(n_b-n)!}}\delta_{n_a-m,n_b-n}|^2}{|s|^{2m}m!n!P_n^{(0,m-n)}(2|s|^2-1)}. \tag{26}$$

From (26) one can see that there exists a constrained condition, $n_a - m = n_b - n$. In particular, when $m = n = 0$, $s = \cosh\lambda$ and then $r = \sinh\lambda$, (26) becomes

$$P(n_a, n_b) = \begin{cases} \sec\mathrm{h}^2\lambda\tanh^{2n_a}\lambda, & n_a = n_b, \\ 0 & n_a \neq n_b, \end{cases} \tag{27}$$

which is just the photon number distribution of the usual two-mode squeezed vacuum state.

In Fig. 1, we plot the distribution $P(n_a, n_b)$ in the Fock space (n_a, n_b) for some given m, n values and parameter λ. It is found from Fig. 1 that the photon number distribution is surely constrained by $n_a - m = n_b - n$. By adding different numbers photons for this two-mode optical fields, we can see that distribution is moved from diagonal line position to off-diagonal line (see Fig. 1(b)). The peak can be translated from zero photons to nonzero photons by adding photons from Figs. 1(a) and (c). Besides, Figs. 1(c) and (d) clearly show that the photon number distribution can also be shifted to the bigger number states and becomes more "flat" and "wide" with the increasing parameter λ.

3.3 Violations of Cauchy-Schwartz Inequality

The cross correlation is always considered for a two-mode system. In this section, to investigate whether the correlations are nonclassical, we evaluate the Cauchy-Schwartz inequality defined as [28]

$$\left(g_1^{(2)}g_2^{(2)}\right)^{1/2} \geq g_{12}^{(2)}, \tag{28}$$

where $g_1^{(2)} = \langle a^{\dagger 2}a^2\rangle/\langle a^\dagger a\rangle$, $g_2^{(2)} = \langle b^{\dagger 2}b^2\rangle/\langle b^\dagger b\rangle$ and $g_{12}^{(2)} = \langle a^\dagger a b^\dagger b\rangle/(\langle a^\dagger a\rangle\langle b^\dagger b\rangle)$. As well known, if this inequality is violated, which means that nonclassical correlation exist. As a measure of this violation, we introduce the function $f_{12}^{(2)}$ defined by

$$f_{12}^{(2)} = \sqrt{\langle a^{\dagger 2}a^2\rangle\langle b^{\dagger 2}b^2\rangle} - \langle a^\dagger a b^\dagger b\rangle. \tag{29}$$

By virtue of the commutation relations $[a, a^\dagger] = 1$ and $[b, b^\dagger] = 1$, (29) can be recast into

$$\begin{aligned}f_{12}^{(2)} &= \sqrt{(\langle a^2a^{\dagger 2}\rangle - 4\langle aa^\dagger\rangle + 2)(\langle b^2b^{\dagger 2}\rangle - 4\langle bb^\dagger\rangle + 2)} \\ &\quad - (\langle aa^\dagger bb^\dagger\rangle - \langle aa^\dagger\rangle - \langle bb^\dagger\rangle + 1).\end{aligned} \tag{30}$$

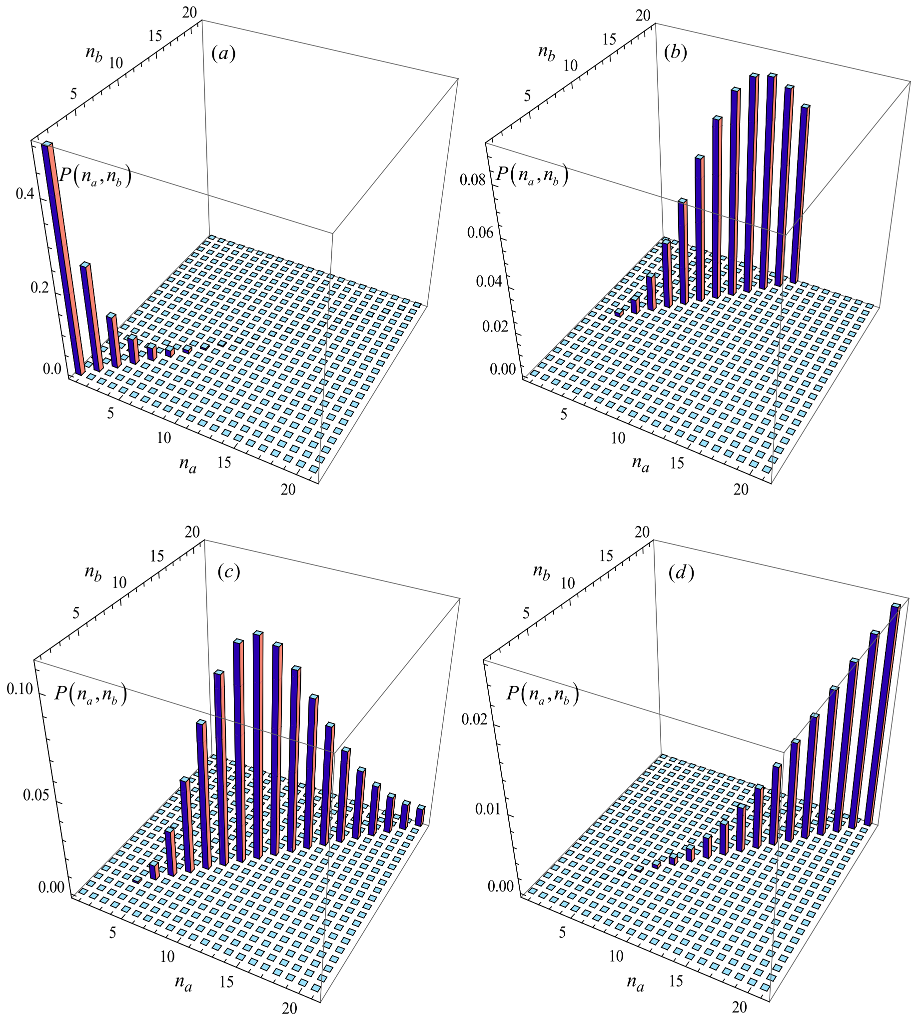

Fig. 1 (Color online) Photon number distribution $P(n_a, n_b)$ in the Fock space (n_a, n_b) for some given m, n values: (**a**) $m = 0, n = 0, s = \sqrt{2}, r = 1$, (**b**) $m = 1, n = 8, s = \sqrt{2}, r = 1$, (**c**) $m = 3, n = 3, s = \sqrt{2}, r = 1$, (**d**) $m = 3, n = 3, s = \sqrt{5}, r = 2$

The value of function $f_{12}^{(2)}$ is negative if inequality in (28) is violated. To obtain the expression of $f_{12}^{(2)}$, in a similar way to (18) and (19), it can be calculated that

$$\langle a^2 a^{\dagger 2} \rangle = |s|^2 (m+2) \frac{P_n^{(0,m-n+2)}(2|s|^2 - 1)}{P_n^{(0,m-n)}(2|s|^2 - 1)}, \tag{31}$$

$$\langle b^2 b^{\dagger 2} \rangle = (n+2) \frac{P_{n+2}^{(0,m-n-2)}(2|s|^2 - 1)}{P_n^{(0,m-n)}(2|s|^2 - 1)}, \tag{32}$$

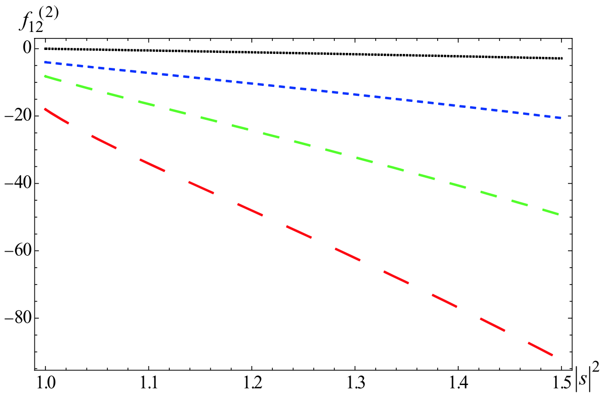

Fig. 2 (Color online) Function $f_{12}^{(2)}$ as a function of $|s|^2$ for the different excitation m, n as follows: $m = n = 0$ (*solid line*), $m = 1, n = 1$ (*dashed line*), $m = 1, n = 3$ (*dotted line*), and $m = 3, n = 3$ (*dash-dotted line*)

and

$$\langle a^\dagger a b^\dagger b\rangle = |s|^2(m+1)(n+1)\frac{P_{n+1}^{(0,m-n)}(2|s|^2-1)}{P_n^{(0,m-n)}(2|s|^2-1)}. \tag{33}$$

In Fig. 2, we display the function $f_{12}^{(2)}$ versus the squeezing parameter $|s|^2$ for the various m, n values for the ETGSVSs. The classical inequality in (30) can be violated corresponding to negative $f_{12}^{(2)}$ for any values of m, n. The bigger squeezing parameter s, the larger violations of Cauchy-Schwartz inequality.

4 Wigner Function of ETGSVSs

The Wigner function is an useful tool to study the nonclassicality of quantum states [29]. In this section, we devote to the research on analytical expression of Wigner function for the state $|r, s, m, n\rangle$. For this propose, we begin with recalling the Weyl ordering of single-mode Wigner operator [30],

$$\Delta_1(\alpha) = \frac{1}{2} \,\vdots\, \delta(\alpha - a)\delta(\alpha^* - a^\dagger) \,\vdots\,, \tag{34}$$

with $\alpha = (q + ip)/\sqrt{2}$ and $\vdots\ \vdots$ denoting Weyl ordering. The merit of Weyl ordering lies in the Weyl ordered operators' invariance under similar transformations [31], i.e.,

$$U \,\vdots\, (\circ\circ\circ) \,\vdots\, U^{-1} = \,\vdots\, U(\circ\circ\circ)U^{-1} \,\vdots\,. \tag{35}$$

Following this invariance and (6), we can see

$$\begin{aligned}
&U_2^\dagger \Delta_1(\alpha)\Delta_2(\beta)U_2(r,s)\\
&= \frac{1}{4}U_2^\dagger \,\vdots\, \delta(\alpha - a)\delta(\alpha^* - a^\dagger)\delta(\beta - b)\delta(\beta^* - b^\dagger) \,\vdots\, U_2(r,s)\\
&= \frac{1}{4} \,\vdots\, \delta(\alpha - sa - rb^\dagger)\delta(\alpha^* - s^*a^\dagger - r^*b)\\
&\quad \times \delta(\beta - ra^\dagger - sb)\delta(\beta^* - r^*a - s^*b^\dagger) \,\vdots
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4} \mathop{:}\limits^{}\, \delta(\bar{\alpha} - a)\delta(\bar{\alpha}^* - a^\dagger)\delta(\bar{\beta} - b)\delta(\bar{\beta}^* - b^\dagger) \,\vdots \\
&= \Delta_1(\bar{\alpha})\Delta_2(\bar{\beta}),
\end{aligned} \tag{36}
$$

where $\bar{\alpha} \equiv s\alpha + r\beta^*$, $\bar{\beta} \equiv r\alpha^* - s\beta$, $\Delta_1(\alpha)$ and $\Delta_2(\beta)$ are single-mode Wigner operators, whose explicit expressions in the coherent state representation are

$$
\Delta_1(\alpha) = e^{2|\alpha|^2} \int \frac{d^2 z_1}{\pi^2} |z_1\rangle\langle -z_1| e^{-2(z_1\alpha^* - \alpha z_1^*)}, \tag{37}
$$

$$
\Delta_2(\beta) = e^{2|\beta|^2} \int \frac{d^2 z_2}{\pi^2} |z_2\rangle\langle -z_2| e^{-2(z_2\beta^* - \beta z_2^*)}. \tag{38}
$$

It can be shown that the Wigner function for the state $|r, s, m, n\rangle$,

$$
\begin{aligned}
W(\alpha, \beta) &= N_{s,m,n}^2 |rs|^{2n} \langle 0, 0| H\left(-isa, -i\frac{b}{r}\right) U_2^\dagger(r, s)\Delta_1 \\
&\quad \otimes \Delta_2 U_2(r, s) H_{m,n}\left(is^* a^\dagger, i\frac{b^\dagger}{r^*}\right)|0, 0\rangle \\
&= N_{s,m,n}^2 |rs|^{2n} e^{2|\bar{\alpha}|^2 + 2|\bar{\beta}|^2} \int \frac{d^2 z_1 d^2 z_2}{\pi^4} e^{-|z_1|^2 - |z_2|^2 - 2(z_1\bar{\alpha}^* - \bar{\alpha} z_1^*) - 2(z_2\bar{\beta}^* - z_2^*\bar{\beta})} \\
&\quad \times H_{m,n}\left(-isz_1, -i\frac{z_2}{r}\right) H_{m,n}\left(-is^* z_1^*, -i\frac{z_2^*}{r^*}\right).
\end{aligned} \tag{39}
$$

Further, employing the generating function of two variables Hermitian polynomials [27],

$$
H_{m,n}(\zeta, \eta) = \frac{\partial^{m+n}}{\partial t^m \partial t'^n} \exp(-tt' + \zeta t + \eta t')|_{t=t'=0}, \tag{40}
$$

(39) can be further rewritten as

$$
\begin{aligned}
W(\alpha, \beta) &= N_{s,m,n}^2 |rs|^{2n} e^{2|\bar{\alpha}|^2 + 2|\bar{\beta}|^2} \frac{\partial^{m+n}}{\partial t^m \partial \tau^n} \frac{\partial^{m+n}}{\partial t'^m \partial \tau'^n} \\
&\quad \times \int \frac{d^2 z_1 d^2 z_2}{\pi^4} e^{-|z_1|^2 - |z_2|^2 - 2(z_1\bar{\alpha}^* - \bar{\alpha} z_1^*) - 2(z_2\bar{\beta}^* - z_2^*\bar{\beta})} \\
&\quad \times e^{-t\tau + (-isz_1)t + (-i\frac{z_2}{r})\tau - t'\tau' + (-is^* z_1^*)t' + (-i\frac{z_2^*}{r^*})\tau'}|_{\tau=\tau'=t=t'=0} \\
&= \frac{1}{\pi^2} N_{s,m,n}^2 |rs|^{2n} e^{-2|\bar{\alpha}|^2 - 2|\bar{\beta}|^2} \\
&\quad \times \frac{\partial^{m+n}}{\partial t^m \partial \tau^n} \frac{\partial^{m+n}}{\partial t'^m \partial \tau'^n} e^{-t\tau - t'\tau' + A^* t' + At + B^*\tau' + B\tau - |s|^2 tt' - \frac{1}{|r|^2}\tau\tau'}|_{\tau=\tau'=t=t'=0}
\end{aligned} \tag{41}
$$

where we have set

$$
A = -2is\bar{\alpha}, \qquad B = \frac{-2i\bar{\beta}}{r}, \tag{42}
$$

and utilized the following integration formula

$$\int \frac{d^2 z}{\pi} e^{\zeta|z|^2+\xi z+\eta z^*} = -\frac{1}{\zeta} e^{-\frac{\xi\eta}{\zeta}}, \quad \mathrm{Re}(\zeta) < 0. \tag{43}$$

Expanding the exponential term $\exp[-|s|^2 tt' - \frac{1}{|r|^2}\tau\tau']$ and using (40), we have

$$\begin{aligned}
W(\alpha,\beta) &= \frac{1}{\pi^2} N_{s,m,n}^2 |rs|^{2n} e^{-2|\bar{\alpha}|^2-2|\bar{\beta}|^2} \frac{\partial^{m+n}}{\partial t^m \partial \tau^n} \frac{\partial^{m+n}}{\partial t'^m \partial \tau'^n} \sum_{k,l=0}^{\infty} \frac{(-|s|^2)^k}{k!} t^k t'^k \frac{(-\frac{1}{|r|^2})^l}{l!} \tau^l \tau'^l \\
&\quad \times \exp(-t\tau - t'\tau' + A^* t' + At + B^*\tau' + B\tau)|_{\tau=\tau'=t=t'=0} \\
&= \frac{1}{\pi^2} N_{s,m,n}^2 |rs|^{2n} e^{-2|\bar{\alpha}|^2-2|\bar{\beta}|^2} \\
&\quad \times \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l}|s|^{2k}|r|^{-2l}}{k!l!} \frac{\partial^{k+l}}{\partial A^k \partial B^l} H_{m,n}(A,B) \frac{\partial^{k+l}}{\partial A^{*k} \partial B^{*l}} H_{m,n}(A^*,B^*) \\
&= \frac{1}{\pi^2} N_{s,m,n}^2 |rs|^{2n} e^{-2|\bar{\alpha}|^2-2|\bar{\beta}|^2} \\
&\quad \times \sum_{k=0}^{m}\sum_{l=0}^{n} \frac{(-1)^{k+l}|s|^{2k}|r|^{-2l}}{k!l!} \frac{m!n!H_{m-k,n-l}(A,B)}{(m-k)!(n-l)!} \frac{m!n!H_{m-k,n-l}(A^*,B^*)}{(m-k)!(n-l)!} \\
&= \frac{1}{\pi^2} N_{s,m,n}^2 |rs|^{2n} e^{-2|\bar{\alpha}|^2-2|\bar{\beta}|^2} \\
&\quad \times \sum_{k=0}^{m}\sum_{l=0}^{n} \frac{(-1)^{k+l}|s|^{2k}|r|^{-2l}(m!n!)^2}{k!l![(m-k)!(n-l)!]^2} |H_{m-k,n-l}(A,B)|^2,
\end{aligned} \tag{44}$$

where the well-known differential relation of $H_{m,n}(\zeta,\eta)$ [27],

$$\frac{\partial^{k+l}}{\partial \zeta^k \partial \eta^l} H_{m,n}(\zeta,\eta) = \frac{m!n!H_{m-k,n-l}(\zeta,\eta)}{(m-k)!(n-l)!}, \tag{45}$$

has been employed. In particular, letting $s=\cosh\lambda$, $r=\sinh\lambda$ and $m=n=0$, i.e., the case for two-mode squeezed vacuum state, (44) becomes

$$\begin{aligned}
W(\alpha,\beta) &= \frac{1}{\pi^2}\exp[-2(|\alpha|^2+|\beta|^2)\cosh 2\lambda - 2(\alpha^*\beta^*+\alpha\beta)\sinh 2\lambda] \\
&= \frac{1}{\pi^2}\exp[-2(x_1^2+p_1^2+x_2^2+p_2^2)\cosh 2\lambda + 2(x_1x_2 - p_1p_2)\sinh 2\lambda]
\end{aligned}$$

which is just the appealing Wigner function of the usual two-mode squeezed vacuum state.

In order to investigate the dependence of the Wigner function for the ETGSVSs on the squeezing parameters s, r and the excitation m, n, we display the Wigner function against $\sigma \equiv \alpha+\beta^*$ and $\gamma \equiv \alpha-\beta^*$ in Fig. 3. It is obvious from Fig. 3(a) to Fig. 3(c) that the more photons we excit, the more peaks the Wigner function appears, while there is some negative region of the Wigner function in the phase space, which is just an evidence of the nonclassicality of the state $|r,s,m,n\rangle$ [32]. In addition, we find that the Wigner function is gathering together with increasing the squeezing parameters r and s (see Figs. 3(c) and (d)).

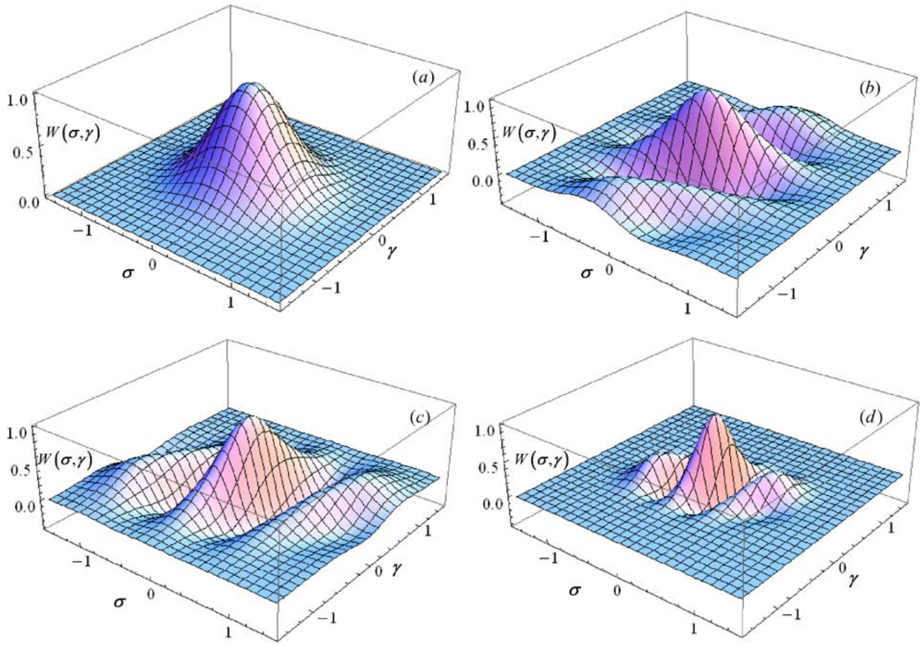

Fig. 3 (Color online) Wigner function of TEGSVS $|r, s, m, n\rangle$ against $\sigma \equiv \alpha + \beta^*$ and $\gamma \equiv \alpha - \beta^*$ for the different squeezing parameters s, r and the different excitation m, n as follows: (**a**) $s = \sqrt{2}$, $r = 1$, $m = n = 0$, (**b**) $s = \sqrt{2}$, $r = 1$, $m = 1$, $n = 3$, (**c**) $s = \sqrt{2}$, $r = 1$, $m = 3$, $n = 1$, (**d**) $s = \sqrt{5}$, $r = 2$, $m = 3$, $n = 1$

5 Conclusions

In summary, we have constructed a new class of excited two-mode generalized squeezed vacuum states (ETGSVSs) $|r, s, m, n\rangle$, which are obtained by repeatedly applying creation operators $a^\dagger$ and $b^\dagger$ on the two-mode generalized squeezed vacuum state. Its normalization constant is also deduced, which is proved to be just a Jacobi polynomial with squeezing parameter s. Then, we find that ETGSVSs can be treated as a generalized squeezed two-variable Hermite polynomial excitation vacuum. Next, squeezing properties and photon number distribution are investigated and the violations of Cauchy-Schwartz inequality are discussed as well, which shows that the bigger squeezing parameter s, the larger violations of Cauchy-Schwartz inequality. Finally, we deduce the explicit expression of the Wigner function for the state $|r, s, m, n\rangle$ by using the Weyl ordered operators' invariance under similar transformations, which is proportional to the two-variable Gaussian-Hermite polynomials, and plot its three-dimensional graphics to observe the behavior strongly depending on the squeezing parameters s, r and the excitation m, n. It is clear that nonclassical effects of the state $|r, s, m, n\rangle$ can remarkably be established due to its negative parts with increasing the values of m and n.

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